...with the best Regards of the A...

AN

INVESTIGATION

OF THE

ERRORS

OF

ALL WRITERS ON ANNUITIES,

IN THEIR

Valuation of Half-yearly and Quarterly Payments,

INCLUDING THOSE OF

SIR ISAAC NEWTON, DEMOIVRE, DR. PRICE, MR. MORGAN, DR. HUTTON,

&c. &c.

With Tables,

SHOWING THE CORRECT VALUES WHEN PAYMENTS ARE MADE IN LESS PERIODS THAN *YEARLY,*

AND A SPECIMEN OF

A Set of Tables on a new Principle,

(NOW IN THE PRESS)

FOR THE VALUATION OF

LEASES, ESTATES, ANNUITIES, CHURCH LIVINGS,

OR ANY INCOME WHATEVER.

BY WILLIAM ROUSE,

Author of the Doctrine of Chances; and Remarks on Freehold and Copyhold Land, Advowsons, &c.

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AN

INVESTIGATION,

&c.

IT may appear a bold assertion for an obscure individual to make, *that all the mathematicians, who have written on the subject* (even including Sir Isaac Newton and Demoivre), *have given erroneous rules and theorems for the valuation of half-yearly and quarterly payments of annuities or incomes*, whether *for life, for years, or for ever*. But truth is equally valuable, from whatever quarter it may flow, and mathematical truths seem to have an advantage over all others; for as they admit of demonstration, they disarm the sceptic, and must be equally received without dispute, both by the enemy and the friend.

When it is considered, that much the greater part of the income of the whole country is received in less *periods* than *yearly*, it must surely be of importance to have correct notions of the difference in the values between such pay-

ments and yearly payments. If the case be applied to the national debt, by supposing the interest 32 millions of pounds per annum, we shall find the following *difference* in the amounts, whether the same be paid *yearly*, and increased at 4 per cent. *per annum*, or paid in 4 quarterly payments of 8 millions each, and increased at 1 per cent. per quarter, being the usual mode of payment.

In 25 years the *difference* will be 31 millions of pounds.
In 50 years the *difference* will be 167 millions of pounds.
In 100 years the *difference* will be 2400 millions of pounds;—and this difference must increase *ad infinitum*.

But, by the theorems and tables now in use, we are taught, that although there is a *difference* between the *values* of two annuities, where one is paid *yearly* and the other *quarterly*, *if they are to continue* 20 *or* 30 *years*, yet if the same annuities are to continue for 100 years, or for ever, there is no *difference* at all!! while, at the same time, it is admitted as self-evident, that the amount and the value must be proportionate to each other.

In tracing these errors, it is curious to observe the inconsistency of each succeeding writer. Sir Isaac Newton and Demoivre began with a *single* error, which increases in every stage of computation, in the same proportion as the *difference* between the yearly and less periods than yearly amounts and values increases; but at the value of the

perpetuity

perpetuity it reaches its maximum, and then it will be in the same proportion to the value of the perpetuity (if paid yearly) that the first year's error was to the first year's value (if paid yearly), as will be seen in CASE I.

Dr. Price and Mr. Morgan, after asserting that this is " *sufficient to prove the fallacy of Mr. Demoivre's* " *method of solution*," actually combine this very error with another of much greater magnitude, and ground theorems (in the same way as Dr. Hutton has done) on such combination of errors, and which NEVER CAN REACH A MAXIMUM. Then Mr. Baily, who says, " he has been par- " ticular in making these observations," having " no where " seen the facts set in their proper point of view;" after animadverting on Dr. Price's rule, and declaring that " the present value of an annuity, payable four times a " year, *must in all cases* be *greater* than when paid only " once a year, " gives an example from his own theorem, to prove, that a perpetual income of four pounds per annum, if the interest be paid *yearly*, is worth £100; but if the interest be paid *quarterly*, it is worth only £98: 10: 3!!!

A like inconsistency appears in a work published under the inviting title of " Sir ISAAC NEWTON'S Tables;" but whoever the author of such part of the Tables might be, it was cruel in him to make Sir Isaac give one rule in page 61, and forget it quite so soon as to begin a table in the very next page, computed in *some* different manner, but *how*,

how, we are left to discover; yet, as the calculator has favoured us with the *amount* of £1 per annum, if improved *half-yearly* at what he calls the same rate (4 per cent. per annum) we are enabled, by comparing them with the amounts, if improved *yearly*, to find the *difference*, and which will always be proportionate to the *difference* of their *values*; for £200, to be received at the end of any number of years, must always be of double the present value of £100, to be *then* received (computing at the same rate of interest.) Now from these tables, in which the editors say, "*they have not* "*been able to discover a single error*," the following is the difference in amounts between yearly and half-yearly payments, at (what is called) 4 per cent. per annum, with *their* difference of values; against which I have stated what should be the *true* difference of values.

No. of Years.	Difference in amounts of yearly and half-yearly payments. £. s. d.	Difference in Values by their Tables. £. s. d.	True Difference of Values. £. s. d.
25	0 12 10	0 1 10	0 4 10
50	3 9 0	0 1 4	0 9 9
100	49 10 0	0 0 4	0 19 7

So that, where the difference in the values should be *greater*, they make it *less*; but what is more singular, they continue their amounts to 150 years (when the difference becomes £532 : 15 : 0, and which requires a difference of £1 : 8 : 3 in the present value to meet it), yet they make but one step in their values from 100 years to eternity,

although

although the adjoining column points out most clearly, that the *proportion* of difference between yearly and half-yearly amounts must *increase ad infinitum;* for this *difference*, which in 25 years is only 1-65th part of the yearly amount, becomes in 50 years the 44th part; in 100 years, the 25th part; and in 150 years it becomes less than the 18th part; and will, in time, *exceed* the yearly amount by an unmeasurable sum. Now can any reasonable man believe, that such part of the tables was the production of Sir ISAAC NEWTON? I admit that the rule given in page 61, is the same as Demoivre's, embracing only a single error, and probably was Sir Isaac's, as they were contemporary, and frequently submitted their calculations to each other.

In Dr. PRICE's work, edited by Mr. MORGAN, there is an Essay which was read to the Royal Society, in the year 1775, containing "Short and easy Theorems for finding, "in all cases, the differences between the Values of "Annuities payable yearly, and of the same Annuities "payable half-yearly, quarterly, or momently;" and it states, "In the foregoing theorems, it may be observed, "that the *ratio* to one another of the values of annuities "payable yearly, half-yearly, quarterly, and momently, is "greatest when *n*" (the time) "is least; that it decreases "continually as *n*" (the time) "increases, till at last it "vanishes when *n*" (the time) "becomes infinite, or the "annuity is a perpetuity." And a note by the Editor states, "The difference, however, between the values of

"annuities

"annuities payable yearly and at shorter intervals, is "known to be continually lessening in proportion to the "length of the term; till at last, when the term is extended "to a perpetuity, those values become the same, whether "the payments are made yearly or momently." And Dr. Hutton, in his Mathematical Dictionary, where he gives a set of theorems for finding the respective values, and a formula for the maximum of *difference*, writes thus, "The difference in the value, by making periods of "payments smaller, for any given term of years, is the "more as the intervals are smaller, or the periods more "frequent. The same difference is also variable, both as "the rate of interest varies, and also as the whole term of "years n varies; and, for any given rate of interest, it is "evident that the difference for any periods m of payments, "first increases from nothing as the term n increases, when "n is 0, to some certain finite term, or value of n, when "the difference D is the greatest, or a maximum; and "that afterwards, as n increases more, that difference will "continually decrease to nothing again, and vanish when n "is infinite."

Now although the above remarks of Dr. Hutton are a little at variance with those of Dr. Price and Mr. Morgan, who assert, that the difference in the values is continually lessening, yet all their theorems indicate a *maximum* of *difference* in twenty-five years.

That

That the generality of writers on annuities (who perhaps seldom investigate rules) should copy theorems authorised by such great names, is not surprising, but it seems extraordinary that a paper, from which such a palpable inconsistency flows, should have escaped the notice of the truly learned and acute members of the Royal Society.

As Zeno's *celebrated* Problem of Achilles and the Tortoise, with Dr. Hutton's reasoning upon it, is applicable, I here give it in his own words.

PROBLEM.

" *If Achilles can walk ten times as fast as a tortoise,*
" *which is a furlong before him, can crawl, will the*
" *former overtake the latter, and how far must he*
" *walk before he does so?*"

" THIS Problem has been thought worthy of notice " merely because Zeno, the founder of the sect of the " Stoics, pretended to prove by a sophism, that Achilles " could never overtake the tortoise; for while Achilles " said he, is walking a furlong, the tortoise will have " advanced the tenth of a furlong; and while the former is " walking that tenth, the tortoise will have advanced the " hundredth part of a furlong, and so on *in infinitum;*

" consequently

" consequently an infinite number of instants must elapse
" before the hero can come up with the reptile, and there-
" fore he will never come up with it.

" Any person however possessed of common sense, may
" readily perceive that Achilles will soon come up with the
" tortoise, since he will get before it."

This is really a parallel case; for the present value of an income to continue any length of time, is just so much money as will come up with, or amount to, the same sum that the income itself will amount to (both being increased by the same rate of interest) at the end of the term. Now, if the amount of £100 per annum, to continue twenty-five years (if received £25 quarterly, and improved at the rate of one per cent, or one-fourth the year's interest quarterly) *exceeds* the amount (when the annuity is received and improved at four per cent yearly,) by nearly 100 pounds, surely the present worth ought to be more in one case than in the other, because it has not only to come up with, or amount to, the same sum that the yearly payments do, but must exceed such sum by nearly 100 pounds; the difference given by their Theorems, (and which is their *maximum* of difference in any period of time) to reach this surplus amount is £13..10..5, but this is at four per cent interest per annum (at which the purchaser's value is limited, otherwise it becomes a new question and a different rate) will not amount to more than £36..1..0.

£36..1..0. In one hundred years, the *difference* of the amounts (if received and improved in the above manner) is upwards of *seven thousand, five hundred pounds*; while the difference in the values given by the theorems, is only £2..15..10, which, in the one hundred years, only amounts to £141, and in three hundred years the *difference* of such amounts is upwards of SIXTY-ONE MILLIONS OF POUNDS; yet according to their theorems, the *difference* of the values now *vanishes!*

As to a *maximum* in the *difference* of values, (when a greater part of an annuity is paid than ought to be, and this is continually increased in a greater ratio than the value is), *such can never take place.* Portions of time are fractions of eternity, and however different men's notions may be of that endless increase of time, yet 50, 100, or 1000 years can only be parts of such a term in all our thoughts; and "any person possessed of common sense, "may readily perceive," that a sum of money thus continually gaining upon another, must, like Achilles, not only come up with the tortoise who moves slower, but get before it, and in time leave it at an unmeasurable distance behind.

Compound interest (or interest upon interest) is grounded on a series of terms increasing in *geometrical* proportion. If £100 be lent at the rate of four per cent. per annum, the lender at the end of the first year would receive £104; and if this be lent for another year, he would receive (at the

the end of the year) a sum bearing the same proportion to this £104, as the £104 did to the £100 first lent. Now, as £100 is to £104, so is £104 to £108..3..2½; and this will be the amount of principal and interest at the end of two years, and so on. Then if £100 be lent for only *half* a year at the same rate of interest *per annum*, (and which all the tables and theorems imply), such a sum should be returned, at the end of the *half year*, as would (being increased in the *same* proportion) amount to £104 at the end of the year; therefore, the lender ought to receive only £101..19..8; for as £100 is to £101..19..8, so is £101..19..8 to £104, and this sum it ought to be at the *end* of the year (and no more) according to the agreed rate of four per cent. *per annum*; indeed, it seems quite as reasonable to make an allowance for the payment of any part of a year's interest before it becomes due, as to give discount for the payment of a bill before it becomes due; and although custom may be pleaded in giving 1-12th part of a year's interest for a month, it is by no means just; but as to questions of compound interest, it is *inadmissible*, for the principal thus becomes increased at a *different* rate *per annum*.

When any sum of money is to be increased in a certain ratio annually, as by 1-20th part, if at 5 per cent., or by 1-25th part, if at 4 per cent.; such *yearly* ratio must be compounded of, and equal to, the *sums* of the *half-yearly*, *quarterly*, or *monthly* ratios, according as the payments

may

may be made in less periods than yearly; that is, if R represents the yearly ratio, or £1 with its interest for a year, then $R^{\frac{1}{2}}$: $R^{\frac{1}{4}}$: $R^{\frac{1}{12}}$ (being the square root, the fourth root, and the twelfth root of R) will respectively represent the half-yearly, quarterly, and monthly ratios, and $R^{\frac{2}{2}}$: $R^{\frac{4}{4}}$: $R^{\frac{12}{12}}$ will each be equal to R itself, and which is the *sum* of their ratios. But this will appear plainer, by the following ratios of £1 at 4 per cent. per annum, whether paid annually or quarterly.

Amount 1st year = R 1·04—Amount 2d year R^2 1·0816

1st Quarter.	2d Quarter.	3d Quarter.	4th Quarter.
$R^{\frac{1}{4}}$	$R^{\frac{2}{4}}$	$R^{\frac{3}{4}}$	$R^{\frac{4}{4}}$
1·009853 :	1·019804 :	1·02985 :	1·04
5th Quarter.	**6th Quarter.**	**7th Quarter.**	**8th Quarter.**
$R^{\frac{5}{4}}$	$R^{\frac{6}{4}}$	$R^{\frac{7}{4}}$	$R^{\frac{8}{4}}$
1·05024 :	1·06059 :	1·07104 :	1·0816

It must be evident on inspection, that any terms compared in each of these series, (being in proportion to yearly and quarterly, or two in the first, and eight in the second), will be the same in amount; for the fourth root of R raised to the eighth power, is the same as R raised to the second power; therefore, whether interest be paid yearly, or (if in a correct manner) quarterly, or momently, the results will be the same, however far the series may be extended.

But,

But, if interest be paid as the common theorems indicate, that is, a *fourth* of the year's interest quarterly, the series will become

Amount if yearly 1·04.—Amount in two years 1·0816.

1st Quarter.	2d Quarter.	3d Quarter.	4th Quarter.
1·01 :	1·0201 :	1·030301 :	1·040604
5th Quarter.	**6th Quarter.**	**7th Quarter.**	**8th Quarter.**
1·05101 :	1·06152 :	1·07213 :	1·08285

It must appear manifest, that no terms in these series will correspond; and the farther they are carried, the *greater* will be the *difference* between the *amounts* of *yearly* and *quarterly* payments; and as all present values must be proportionate to their respective amounts, if such *differences* continually increase, so will the *differences* in the respective values continually increase.

An annuity is nothing more than a yearly interest on a corresponding sum of money; for an annuity of £1, and the interest of £25, if at 4 per cent., or of £20, if at 5 per cent., amount to one and the same sum; therefore, if an annuity is to be divided into quarterly payments, each quarter's payment should be in the same ratio to the true *quarter's* interest, that the annuity is to the true *yearly* interest. Now the true yearly interest of £1 (the annuity) at 4 per cent., is ·04; therefore, as ·04 is to 1 (the annuity), so

so is ·0098533 (the true quarter's interest) to ·2463325 (the true quarter's payment), being 4*s.* 11*d.* instead of 5 shillings.

Whether equal sums of money, paid periodically, are called interest or annuities, their increase will be the same; and if *correct* portions of such sums are paid each *time*, and these are increased according to the *true ratio;* their *amounts* and their *values* will increase in the same proportion, for any number of years, or for ever, whether paid and increased *yearly* or *momently;* but if the *payments*, or the *ratio*, or *both*, be wrong, the *amounts*, in every period of time, must be different from the *yearly amounts*, as will be clearly seen by the following four cases.

CASE I.

Where the part of the Yearly Sum, or Annuity, is WRONG; *but the ratio of the Rate of Interest operating upon it, is* RIGHT.

THIS Case supposes ·25, or $\frac{1}{4}$ of the annual sum to be paid *quarterly*, but improved in the correct ratio 1.0098533; and as this difference between the true payment ·2463325,

and

and ·25, will be the same in each quarter, while the ratio of the rate of interest operating upon it, is *correct*, whatever proportion the *surplus* interest or amount in one year bears to the *correct* interest or amount for one year, if the *like proportion* be *added* to any other correct *yearly* value, in any stage of time, it will give the *correct present value*, if the annuity and interest be paid quarterly in this manner; for example,—

If each quarter's payment be ·2463325, and improved in the ratio 1·0098533, at the end of the year the amount will be precisely £1, being the true amount of the first year's payment; but if each quarter's payment be ·25, and improved in the same ratio, the amount at the end of the year will be 1·014888: and in the same proportion as this exceeds 1, so will the true value for *quarterly* payments (at 4 per cent. per annum) exceed the value for *yearly* payments at the same rate of interest per annum. Now the present value of £1, to be received at the end of the year, is in the tables (at 4 per cent.) ·961538; therefore, as £1 is to £1·014888, so is £·961538 to £·975854, the true value, if the £1 is to be made in four quarterly payments of five shillings each: and as £1 is to £1·014888, so will £25, the value of the perpetuity, if paid yearly, be to £25·37221, the true value to give the purchaser 4 per cent. per annum, if the same be paid and improved in this manner quarterly; and a like reasoning will apply to the *yearly* values for any intermediate stage of time. The proportion

of

of this excess is about the 67th part of the yearly value, and always continues the same.

This is the error of both Sir Isaac Newton and Demoivre; their ratio of the rate of interest was *right*, but the part of the annuity paid was *wrong*; yet as this *excess* of annuity is constantly improved by a correct ratio of interest, such annual *excess* must require 25 years purchase for the value of the perpetuity, equally with the annuity itself; and this is precisely the result, for £1·014888 is rather more than £1..0..3½, and £25·37221, or £25..7..5¼ is 25 times the annual payment, as it ought to be. Indeed it must be evident to common sense, that an income of £1 : 0 : 3½ per annum, requires more purchase money than £1 per annum, supposing the *same* rate of interest in both cases, which all the tables and theorems imply.

CASE II.

Where the part of the Annuity paid is RIGHT, *as* ·2463325 *quarterly; but the Ratio of the Rate of Interest is* WRONG, *being* 1·01 *quarterly, instead of* 1·0098533.

Here, the surplus interest, or amount, does not continue each succeeding year in the same *proportion* to the respec-

 tive

tive *yearly* amount that it bears the first year, but is continually increasing; for this *difference*, which is not the 4000th part of the yearly amount, or £1 the first year, becomes in 50 years the 52d part of the yearly amount in that time; in 100 years, it will become the 22d part; and in 500 years, this *difference*, between paying and improving only £1 per annum, *yearly*, at 4 per cent. or *quarterly* in the above manner, amounts to more than *two thousand six hundred millions of pounds!!* which is nearly the third part of the whole *yearly* amount; indeed, in 1200 years, the *difference alone* becomes equal to the *whole yearly amount* in the time, and beyond this term it exceeds the *amount* (improved yearly at 4 per cent.), and will go on increasing, till such *excess* becomes an unmeasurable sum. This case implies £1 per annum (or any annuity), improved in the ratio 1·040604, or about £4..1..2½ per cent. per annum, which is as distinct from 4 per cent. as from 14; and in proportion as such ratio increases any annual sum more rapidly, so must the amount of any *present worth* be increased (if improved at only 4 per cent. per annum) to meet it.

CASE

CASE III.

Where the part of the Annuity and the Ratio of the Rate of Interest are BOTH WRONG ; *the Payment being One Quarter of the yearly Sum, or* ·25 *instead of* ·2463325, *and the Ratio being* 1·01, *instead of* 1.0098533.

THE theorems of Dr. Price, Mr. Morgan, Dr. Hutton, &c. are formed on this *combination of errors*, which will account for the great and increasing *differences* between amounts, when improved *correctly*, or according to their theorems.

The *differences* arising in this case will be the *sum* of the differences of the first and second cases ; for, instead of £1 per annum, increased in the ratio 1·04 (or 4 per cent. per annum), it becomes £1..0..3½ per annum, increased in the ratio 1·040604, or £4..1..2½ per cent. per annum.

The amount of £1 per annum, thus received and improved quarterly for 100 years, will exceed the amount, if correctly received and improved at the rate of 4 per cent. per annum, by nearly 1-22^d part of the *yearly* amount (as in Case II), *added* to nearly 1-67^th part (as in Case I) ;

 for

for the amount, if thus received and improved quarterly for 100 years, is £1313;—but the amount of £1 per annum, if improved at 4 per cent. per annum, for 100 years, is only £1237 (leaving out decimals), and the difference, £76, will be very nearly 1-22[d] part, *added* to 1-67[th] part of £1237, and so is the difference in the values; for according to the tables, the value of £1 per annum for 100 years, at 4 per cent., is £24·5050, but this is as an equivalent for what will amount to £1237, in the time: if the annuity be received and improved quarterly, agreeably to this case, the amount will be £1313, the true value of which, by the tables now given, is £26·0003, and the purchaser will make 4 per cent. per annum of his money, as much in the one case as in the other. If an annuity be received and improved in this manner quarterly for 1200 years, or for 4800 quarters, it will amount to more than *double the sum* that the same annuity will amount to in the same time, if paid and improved correctly at 4 per cent. per annum, and of course will be worth *twice* the present value of the other. It may appear extraordinary at first view, that the perpetuity requires but 25 years purchase, (however often received and improved, if it be *correctly* received and improved), and yet, *apparently*, this same annuity for a finite term requires upwards of 50 years purchase, which *seems* only to pay 2 per cent. instead of 4 per cent.

Probably such a first impression induced Mr. Morgan to reject

reject Demoivre's rule; for Mr. M. says, "According to "Demoivre's rule, if the term be extended, an annuity "payable quarterly will be worth more than even the "*perpetuity* when the payments are made yearly. This "appears to be very erroneous, and sufficient to prove the "fallacy of Demoivre's method of solution." But this very circumstance is "sufficient to prove" the *truth* of Demoivre's method of solution (as far as it goes); for it must be evident to common sense, that more money is required to purchase £4..1..2½ per annum for ever (or even for a finite term, if extended), than to purchase £4 per annum for ever, supposing the same rate in both cases, which is always presumed; but if two equivalent sums are improved by different rates of interest, the longer the term, the more must they be separated; and the circumstance of the value being 50 years purchase for an annuity (if paid quarterly in this manner), and yet give the purchaser 4 per cent. per annum interest, when duly considered, will excite no more astonishment than for a man to employ his capital at 4 per cent. interest per annum, who may pay £500 for only five shillings a year for 30 or 40 years, with a valuable reversion at the end of the term. So it is in the above case, for £1 per annum (as it is called), if paid at five shillings per quarter, and improved in the ratio of 1 per cent. per quarter, will, in 1200 years, amount, not only to the same sum that the £1 per annum will, if paid yearly and improved at 4 per cent., but precisely to *as much more* as the extra £25 will amount to in the same time, if improved in the ratio of 4 per cent. per annum.

CASE

CASE IV.

Where the part of the Annuity, and the Ratio of the Rate of Interest, are BOTH RIGHT ; *the Quarterly Payment being* ·2463325, *and improved in the Ratio* 1·0098533.

WHEN a just proportion is paid of the annuity, and it is increased in a correct ratio, according to the number of payments in a year, there will be no difference in any corresponding period of time, between the yearly or momently amounts and values.

If the value be required of £1 per annum for 50 years, so as to pay the purchaser 4 per cent. per annum interest, let the annuity, or £1, be equal to u, 1·04 (or £1 with a years interest) $= R$, 50 years (or the time) $= t$, then

$$\frac{u - \frac{u}{R^t}}{R-1}, \text{ or } \frac{£1 - \frac{1}{1{\cdot}04\rangle^{50}}}{{\cdot}04} = \text{value; being £21·4822 for}$$

the yearly payments ; and if the payments are to be made, m times in a year, the m^{th} root of R, or $R^{\frac{1}{m}}$, will express the ratio of the rate of interest for the *time*, and $R^{\frac{1}{m}} - 1$ will be the correct interest for the *time*. It has been shown that the true part of the annuity for the time will be in the same proportion to the true interest, that the annuity is to the year's interest, and calling this part of the annuity

q,

q, it will be found thus: As $R-1 : £1 :: R^{\frac{1}{m}}-1 : \frac{R^{\frac{1}{m}}-1}{R-1} \times £1 = q$, this is merely dividing the ***true interest for the time*** by the *year's interest* (as multiplying by £1 makes no alteration); for if the payments be quarterly, $R^{\frac{1}{m}}-1$, will be ·0098533, and this divided by $R-1$, or ·04, gives ·2463325 for q, or the true quarter's payment; and if the payments be monthly, $R^{\frac{1}{m}}-1$ will be ·0032737, which, divided by ·04, gives ·0818425 for q, or the true monthly payment. The value of the annuity for t years, if the payments be made m times in a year, will be thus found:

$$\frac{q - \frac{q}{R^{\frac{mt}{m}}}}{R^{\frac{1}{m}}-1} = \text{value;}$$

or if quarterly for 50 years,

$$\frac{\cdot2463325 - \frac{\cdot2463325}{\overline{1\cdot04}|^{\frac{200}{4}}}}{\cdot0098533} = £21\cdot4822, \text{ the same as}$$

the yearly value. R raised to t years, is the same as raising the m^{th} root of R to m times t years; therefore, $\overline{1\cdot04}|^{50}$ may be used instead of $\overline{1\cdot04}|^{\frac{200}{4}}$. If the annuity is to be perpetual, here t, or the time, becomes infinite, and whatever sum is supposed to be divided by an *infinite* quantity, the quotient must be equal to nothing, or 0; therefore $\frac{u}{R^t}$ in the *yearly* payments, and $\frac{q}{R^{\frac{mt}{m}}}$ in the m payments in a year, becoming equal to 0, may be left out; and

and the other part $\frac{u}{R-1}$, being merely the annuity, or yearly sum, divided by the interest for a year, will give the value of the perpetuity; as $\frac{£1}{\cdot 04} = £25$ for the value of the perpetuity, if in *yearly* payments; and in the theorem for m payments in a year $\frac{q}{R^{\frac{1}{m}}-1}$, or $\frac{\cdot 2463325}{\cdot 0098533}$ $= £25$ (the same as the *yearly* value), being the *correct* part of the annuity divided by the *correct* part of the interest.

It may be proper to remark here, that the *arithmetical* part of the annuity, or £·25, divided by a like *arithmetical* part of the yearly interest, or £·01 (as by the common theorems), will give the same quotient, or £25; and such may have led to the opinion, that "when the term is "extended to a perpetuity, those values become the same, "whether the payments are made yearly or momently:" but this by no means proves it; for *any part* of *any other dividend*, if divided by a *like part* of its divisor, the quotient will be the same as the *dividend itself*, divided by the *divisor itself:* 84 divided by 12, gives 7; and so will $\frac{1}{4}$ of 84 (or 21) if divided by $\frac{1}{4}$ of 12 (or 3). Indeed the above proves (what is named in Case III.) that the purchaser, instead of making 4 per cent. per annum, makes £4..1..2$\frac{1}{2}$ per cent. per annum interest; for ·01 is the *correct* quarterly payment of such a rate of interest per annum; and £·25 is the *correct* quarterly part of £1..0..3$\frac{1}{2}$ per annum; therefore he gives £25 for

for a perpetual income (as near as money can mark it), of £1..0..3½ per annum. The value of 5*s*. per quarter for ever (if improved as in Case I.), to give the purchaser 4 per cent. interest per annum, is £25..7..5¼. If the purchaser were allowed 5 per cent. per annum interest, such a perpetual income would be worth only £20..7..4¾. As the interest allowed is no more than £4..1..2½ per cent. per annum, the value comes out £25 precisely, as it ought to do; but this is as totally distinct from any question of 4 per cent., as it is from 14 per cent. interest per annum.

In all cases, for a finite term of years, the most simple rule to ascertain the true value, will be—*First to find the* AMOUNT *of the annuity at the end of the term* (whether paid yearly, or divided into half-yearly, quarterly or momently payments), *according to the ratio by which it is increased each* TIME *of payment* (without considering the per centum *per annum*): *then*, (whatever interest per annum is to be allowed the purchaser) *raise R* (or £1 with the interest to be allowed for a year), *to the power denoted by the given number of years, and divide the* AMOUNT (first found) *by this sum, the quotient will be the true value*, so as to give the purchaser the rate per cent. per annum agreed upon.

The *amount* at the end of the term, however often the payments are made, will be found by the following theorem.

Call

Call the number of payments to be made, t.

Each payment .... u.

£1 with its interest for one *time* ...... r.

then $\frac{r^t-1}{r-1} \times u =$ amount. And if the amount of £25 per quarter, improved at 1 per cent. per quarter, for 100 years, were required, we have $t=400$; $u=$ £25; and $r=1{\cdot}01$. Therefore, $\frac{\overline{1{\cdot}01}|^{400}-1}{{\cdot}01} \times$ £25 = amount, being £131313·5.

And to find the value, so as to give the purchaser 4 per cent per annum interest, raise $R=1{\cdot}04$ (being £1 with a year's interest), to the hundredth power, for a divisor; then £131313·5, divided by $\overline{1{\cdot}04}|^{100}$, or 50·5045, gives £2600, for the true value of £25 per quarter, supposing it improved at 1 per cent. per quarter for 100 years; so as to give the purchaser 4 per cent. interest per annum.

By applying the foregoing theorem to the dividends on the national debt, calling them 32 millions per annum, the *great difference* between paying them in the usual manner, of one-fourth part every quarter, and supposing them increased in the ratio of one-fourth the yearly interest quarterly, or paying them as the nature of the contract seems to imply: viz. yearly (or correctly, if otherwise), will appear as follows:

The amount in 25 years, if paid in four quarterly sums, of eight millions each, and improved each quarter by one-fourth

fourth of a year's interest, or £1 per cent. per quarter, will be

$$\frac{\overline{1{\cdot}01}|^{100}-1}{{\cdot}01} \times £8000000 = £1363864000.$$

But if paid and increased annually, or by a *true* part, both of annuity and interest, (if paid quarterly), the amount will be

$$\frac{\overline{1{\cdot}0098533}|^{100}-1}{{\cdot}0098533} \times £7882640 = £1332664000,$$

making a difference in the amounts, in 25 years, of *thirty-one millions two hundred thousand pounds*. Proceeding in the same manner, the difference will be found in 50 years to be more than 167 millions; and in 100 years, to be upwards of TWO THOUSAND FOUR HUNDRED MILLIONS OF POUNDS.

The following table shows, at one view, the precise parts of £1, that ought to be paid, if the annuity be divided into half-yearly, quarterly, or monthly payments, according to the rates of 4 or 5 per cent. per annum interest.

	4 per Cent. *per Annum.*	5 per Cent. *per Annum.*
Yearly payments	£1·	£1·
Half-yearly ditto	·4951000	·4939200
Quarterly ditto	·2463325	·2454600
Monthly ditto	·0818425	·0814880

This Table is very readily formed, for an annuity of £1, being the same as one year's interest on £25 (if at 4 per

4 per cent.), or on £20 (if at 5 per cent.), it is merely multiplying the *true* interest on £1, for the *time*, by 25 or by 20, as the case may be. The use of this table will appear by the following examples:—

If a person enjoy an income of £100 per annum, and be desirous of being paid quarterly; what ought he to receive each quarter, computing at 4 per cent. interest per annum? The true part of £1 per annum being ·2463325, it is merely multiplying this decimal by the annuity or £100, which gives £24·63325, or £24..12..8 is the true quarterly payment.

In the same manner, the true quarterly payment of interest on the national debt, supposing it 32 millions per annum, is found by multiplying the above decimal by 32 millions, which gives £7882640, instead of 8 millions, making a difference of £117360 per quarter.

If a person enjoy an income of £600 per annum, and be desirous of being paid monthly, or 12 times a year; what ought to be paid each month, computing at 5 per cent. interest per annum? Here, the true part of £1 per annum for a month is ·0814880, which multiplied by £600, gives £48·8928000, or £48..17..10¼ for the true monthly payment. And if any income, receivable in *less periods* than yearly, is to be exchanged for an equivalent *yearly* payment, it will be only reversing the operation; for if a

monthly

monthly income of £48·8928000 is to be exchanged for an equivalent yearly income, divide the monthly income by the true monthly part of £1 per annum, or by ·0814880, and the answer will be £600, as above.

The following Tables show at one view the amount of £1 per annum, for any number of years therein named, whether paid and improved annually, or paid in the usual manner of 10*s.* half yearly, or 5*s.* quarterly, and increased in the ratio of *one half*, or *one quarter*, the annual interest; also the present value of the same, so as to give the purchaser 4 or 5 per cent. interest per annum, as named in its respective column.

TABLE I.

Amounts and Values of Half-Yearly Payments, at (what is called) 4 per Cent. per Annum.

Amount of £1 per annum, if paid and improved *yearly*, at 4 per Cent.		Present value of £1 per annum, if paid and improved *yearly*, at 4 per cent.		N° of Years.	Amount of ten shillings, if paid and improved *half-yearly*, at 2 per Cent.		Present value of ten shillings, if paid and improved half-yearly, at 2 per Cent. so as to give the purchaser 4 per Cent. per annum Interest.	
£	Decimal Parts.	£	Decimal Parts.		£	Decimal Parts.	£	Decimal Parts.
1	·0000	0	·9615	1	1	·0100	0	·97115
2	·0400	1	·8861	2	2	·0607	1	·9053
3	·1216	2	·7751	3	3	·1540	2	·8039
4	·2464	3	·6299	4	4	·2915	3	·6684
5	·4163	4	·4518	5	5	·4750	4	·5005
12	·0061	8	·1109	10	12	·1487	8	·2072
20	·0236	11	·1184	15	20	·2841	11	·2631
29	·7781	13	·5903	20	30	·2112	13	·7880
41	·6459	15	·6221	25	42	·2900	15	·8636
56	·0849	17	·2920	30	57	·0260	17	·5822
95	·0255	19	·7927	40	96	·8870	20	·1805
152	·6671	21	·4822	50	156	·1175	21	·9678
237	·9907	22	·6235	60	244	·1309	23	·2072
364	·2905	23	·3945	70	374	·9150	24	·0770
551	·2450	23	·9154	80	569	·2537	24	·6968
827	·9833	24	·2673	90	858	·0308	25	·1481
1237	·6237	24	·5050	100	1287	·1407	25	·4856
63742	·5882	24	·9902	200	68843	·3809	26	·9899
8214667924	·5283	24	·9999	500	9957263636	·3636	30	·3033

TABLE II.

Amounts and Values of Quarterly Payments, at (what is called) 4 per Cent. per Annum.

Amount of £1 per annum, if paid and improved *yearly*, at 4 per cent.		Present value of £1 per annum, if paid and improved *yearly*, at 4 per cent.		No. of Years.	Amount of five shillings if paid and improved *quarterly*, at 1 per Cent.		Present value of five shillings, if paid and improved *quarterly*, at 1 per Cent, so as to give the Purchaser 4 per Cent per annum Interest.	
£	Decimal Parts.	£	Decimal Parts.		£	Decimal Parts.	£	Decimal Parts.
1	·0000	0	·9615	1	1	·0151	0	·97606
2	·0400	1	·8861	2	2	·0714	1	·9151
3	·1216	2	·7751	3	3	·1706	2	·8187
4	·2464	3	·6299	4	4	·3145	3	·6880
5	·4163	4	·4518	5	5	·5047	4	·5245
12	·0061	8	·1109	10	12	·2217	8	·2565
20	·0236	11	·1184	15	20	·4176	11	·3372
29	·7781	13	·5903	20	30	·4181	13	·8824
41	·6459	15	·6221	25	42	·6207	15	·9878
56	·0849	17	·2920	30	57	·5102	17	·7315
95	·0255	19	·7927	40	97	·8467	20	·3804
152	·6671	21	·4822	50	157	·9026	22	·2190
237	·9907	22	·6235	60	247	·3174	23	·5101
364	·2905	23	·3945	70	380	·4449	24	·4321
551	·2450	23	·9154	80	578	·6536	25	·1047
827	·9833	24	·2673	90	873	·7604	25	·6091
1237	·6237	24	·5050	100	1313	·1350	26	·0003
63742	·5882	24	·9902	200	71599	·2300	28	·0704
8214667924	·5283	24	·9999	500	10983475000	·0000	33	·4264

TABLE III.

Amounts and Values of Half-yearly Payments, at (what is called) 5 per Cent. per Annum.

Amount of £1 per annum, if paid and improved *yearly*, at five per cent.		Present value of £1 per annum, if paid and improved *yearly* 5 per cent.		No. of Years.	Amount of ten shillings, if paid and improved *half-yearly*, at 2 and a half per cent.		Present value of ten shillings, if paid and improved half-yearly, at two and a half per cent, so as to give the purchaser 5 per cent. per annum interest.	
£	Decimal Parts.	£	Decimal Parts.		£	Decimal Parts.	£	Decimal Parts.
1	·0000	0	·9524	1	1	·0125	0	·96428
2	·0500	1	·8594	2	2	·0762	1	·8832
3	·1525	2	·7232	3	3	·1938	2	·7589
4	·3101	3	·5459	4	4	·3680	3	·5936
5	·5256	4	·3295	5	5	·6017	4	·3891
12	·5779	7	·7217	10	12	·7724	7	·8411
21	·5786	10	·3796	15	21	·9514	10	·5590
33	·0660	12	·4622	20	33	·7014	12	·7017
47	·7271	14	·0939	25	48	·7424	14	·3938
66	·4388	15	·3724	30	67	·9962	15	·7328
120	·7998	17	·1591	40	124	·1922	17	·6409
209	·3480	18	·2559	50	216	·2761	18	·8601
353	·5837	18	·9293	60	367	·1666	19	·6564
588	·5285	19	3·427	70	614	·4183	20	·1936
971	·2288	19	·5965	80	1019	·5705	20	·5718
1594	·6073	19	·7522	90	1683	·4600	20	·8528
2610	·0252	19	·8479	100	2771	·3210	21	·0744
345831	·7600	19	·9988	200	389554	·0000	22	·5272
786466000000	·0000	19	·9999	500	1059082324455	·0000	26	·9327

TABLE IV.

Amounts and Values of Quarterly Payments, at (what is called) 5 per Cent. per Annum.

Amount of £1 per annum, if paid and improved *yearly*, at 5 per cent.		Present value of £1 per annum, if paid and improved *yearly*, at 5 per cent.		Nº of Years.	Amount of five shillings, if paid and improved *quarterly*, at 1 and 1-4th per cent.		Present value of five shillings, if paid and improved quarterly, at 1 & 1-4th per cent. so as to give the purchaser 5 per cent. per annum interest.	
£	Decimal Parts.	£	Decimal Parts.		£	Decimal Parts.	£	Decimal Parts.
1	·0000	0	·9524	1	1	·0189	0	·97028
2	·0500	1	·8594	2	2	·0897	1	·8954
3	·1525	2	·7232	3	3	·2150	2	·7773
4	·3101	3	·5459	4	4	·3978	3	·6180
5	·5256	4	·3295	5	5	·6407	4	·4196
12	·5779	7	·7217	10	12	·8723	7	·9025
21	·5786	10	·3796	15	22	·1434	10	·6513
33	·0660	12	·4622	20	34	·0294	12	·8253
47	·7271	14	·0939	25	49	·2676	14	·5488
66	·4388	15	·3724	30	68	·8034	15	·9195
120	·7998	17	·1591	40	125	·9587	17	·8919
209	·3480	18	·2559	50	219	·9000	19	·1761
353	·5837	18	·9293	60	374	·3027	20	·0385
588	·5285	19	·3427	70	628	·0816	20	·6426
971	·2288	19	·5965	80	1045	·1966	21	·0889
1594	·6073	19	·7522	90	1730	·7740	21	·4389
2610	·0252	19	·8479	100	2857	·6000	21	·7306
345831	·7600	19	·9988	200	414008	·1900	23	·9413
786466000000	·0000	19	·9999	500	1233190571428	·0000	31	·3603

By these tables, the amounts and values of any other *yearly*, *half-yearly*, or *quarterly payments*, or gross *sums;* also the value of *reversions*, or *renewal of leases*, may be readily found, as by the following examples:—

What will be the amount and value of a *yearly* income of £24, improved at 5 per cent. per annum, to continue for 80 years?

In the first column of Table IV. the amount of £1 per annum, if paid and improved *yearly*, is, for 80 years, £971·2288; therefore, 24 times this sum, or £23309·4912, equal to £23309 .. 9 .. 9 will be the amount, and 24 times £19·5965 (in the next column), or £470 .. 6 .. 3 is the present value.

What will be the amount of a *quarterly* income of £6, improved at $1\frac{1}{4}$ per cent. per quarter, to continue for 80 years? and what is the present value, so as to give the purchaser 5 per cent. per annum interest?

This being 24 times the quarterly income, or five shillings, at the head of the fourth column in the same table, we must multiply £1045·1966 (against 80 years), by 24, producing £25084·7184, or £25084 .. 14 .. 4 for the amount; and 24 times £21·0889, being £506·1336, or £506 .. 2 .. 8, is the value; so as to give the purchaser 5 per cent. per annum interest.

What

What will £500 amount to in 80 years, at 5 per cent. per annum interest?

As each line in the first and fourth columns shows the *amount* of its respective line in the adjoining or second and fifth columns, *either* may be used. By the rule of proportion, if £21·0889 amount to £1045·1966, £500 will (in the same time, and at the same rate) amount to £24780·918, or £24780 .. 18 .. 4. And if the value of a *reversion* were required, it would be only reversing the operation; for, if £1045·1966 (to be received at the end of 80 years) require (a present value of) £21·0889, the sum of £24780·918 (to be *then* received) will require £500 in present money.

The renewal of any lapsed term in a lease (whether for years or for lives), is nothing more than paying the *difference* between the value of a new lease for the original term, and the value of the unexpired term in the old lease.

If the original term were 40 years, and 15 years are unexpired, the *difference* between the value for 40 years, and the value for 25 years (the unexpired term) will be the value for renewal; the value of £1 per annum, for 40 years, at 5 per cent. if paid *yearly*, is £17·1591, and for 25 years, is 14·0939; the difference, or 3·0652 years' purchase, is the value for renewal. But if the rent be paid *quarterly*, the *difference* between the values for 40

 and

and 25 years, in the fifth column, will be *too much;* for the amounts in the fourth column (and by which the fifth is formed) increase in a greater ratio than 5 per cent. per annum; yet, as 10·6513, against 15 years, is the correct value for 15 years *(when the 25 years are expired)*, if paid quarterly, we must find what present sum will, in 25 years, amount to £10·6513, which is readily done by the rule of proportion; for if £49·2676 (to be paid in 25 years) require £14·5488 in hand, £10·6513 (to be *then* paid) will require £3·1453; or so many years' purchase is the value for renewal, if the rent be paid in the usual manner, *quarterly.*

As to the tables published at rates of interest *above* 5 per cent. per annum, when the principle on which they are formed is considered, they will be found both impracticable and illusive to purchasers.

The principle on which all the tables hitherto published, for the valuation of terminable incomes, whether for years or lives, have been formed, is, that the yearly income will not only be equal to the interest per cent. named in the tables, but as *much more* as will replace (at the end of the term or life) the capital employed: for instance, if a person pay £802 for an income of £100 per year, for 17 years, he employs his money, or capital, at 10 per cent. interest (according to the Tables), that is, he is supposed to receive £10 for every £100 advanced, and as much more as will replace

replace the £802 at the end of the 17 years. Ten per cent. on the capital employed is £80..4..0, but as he will receive £100 each year, the difference, or £19..16..0 per year, is the sum to replace the £802 at the end of the term; this it certainly will do, *if* a man can make 10 per cent. interest on £19..16..0 every time he receives it; but at 5 per cent. the same sum will only amount to £511 in 17 years. Now I appeal to the common sense of every man, if it be practicable to improve small sums of money at a greater interest than 5 per cent.?—indeed, beyond this rate it is illegal to lend money; and no leases, annuities, or government securities, can be purchased with small sums of a few pounds each, which in general form the excess of interest, to replace the capital with, when the income ceases.

Such being the principle on which all the tables of compound interest, for the valuation of leases and annuities for years or lives are formed, they must be *practically* wrong, where they exceed the rate of 5 per cent. which being the legal interest of the country, all calculations to replace the capital at the end of the term, ought to be made in this rate; *and as these tables form the basis of all the calculations for annuities on lives*, they must follow the same fate: for the present value of an annuity *certain* for any number of years, is the several *present* values of the several sums to be received at the end of the first, second, third, &c. years, to the end of the term, *added together;* and the

present

present value of a life annuity is nothing more than the amount of the said several values, *each diminished* in proportion as the respective probabilities of the person being alive at the end of the several years to receive them, are below certainty, and continued to the utmost probable extent of life; this, according to Demoivre's hypothesis, is eighty-six years, and, according to Price and others, is twice the expectation found by the tables formed to show the progression of mortality, which greatly varies in different countries, and indeed in the same country at different periods; for the continual discoveries in medical philosophy to lessen the mortality of diseases, seem to have rendered the tables formed fifty or one hundred years ago too incorrect for the calculations of the present time. A French author has computed, that the mean duration of human life is increased, at least *three years*, by the vaccine inoculation alone. Now, as the values of annuities for lives depend on the combinations of these two sets of tables, if one requires new modelling, and the other is practically wrong, all the results at rates exceeding 5 per cent. must be doubly incorrect.

The following is a specimen of a set of Pocket Tables, (which are now in the press), showing at one view, what interest any purchaser makes of his money in buying leases, annuities, or any nett yearly income whatever; and so that the returning sums to replace the capital when the income expires, are computed at 5 per cent. interest (which

may

may be practicable); also showing what *Freehold*, or *perpetual Income*, is of equal value with the purchase in question.

Number of years to continue.	Yearly income for each £100 paid.	Interest per cent. for the time and capital replaced.	A Freehold of equal Value.
	£	£ s. d.	£ s. d.
10	15	7....1....0	5...15...10
10	16	8....1....0	6....3....6
15	11	6....7....3	5...14....2
15	13	8....7....3	6...14...11
20	9	5...19....6	5...12....2
20	12	8...19....6	7....9....7
25	8	5...18....1	5...12....9
25	11	8...18....1	7...15....0
30	7	5....9...10	5....7....7
30	10	8....9...10	7...13....9
50	6	5...10....5	5....9....6
50	10	9...10....5	9....2....7

EXAMPLE.

If I give £100 for £16 per annum to continue for 10 years; the first column shows the number of years (or 10); the second, the yearly income for each £100 paid (or £16), against which, in the 3d column, is £8..1..0, being the rate of interest the money is vested at for the time; and the

4th

4th column shows that such an investment is equal to the purchase of a freehold, or perpetual income of £6..3..6 per annum, both of which are thus proved: the difference between the interest £8..1..0, and the yearly income £16 is £7..19..0 per annum; now this improved at 5 per cent. per annum, will amount in 10 years (when the income expires) to £100, being the purchase money; and the difference between £6..3..6 in the 4th column, and the £16 received, is £9..16..6 per annum, which, improved at 5 per cent. will in 10 years amount to £123..10..0, being sufficient to purchase a perpetual income of the same amount (£6..3..6) when the 10 years have expired.

On this simple principle the Tables have been formed; they are extended to 100 years, and will most readily answer for any rate of interest required. They will be accompanied with Tables of simple and compound interest; true and customary discounts, values of annuities on lives, church livings, &c. with a short history of interest, and an elucidation of the principles on which the valuation of life annuities is grounded.

FINIS.

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